That's a State Space Model too!

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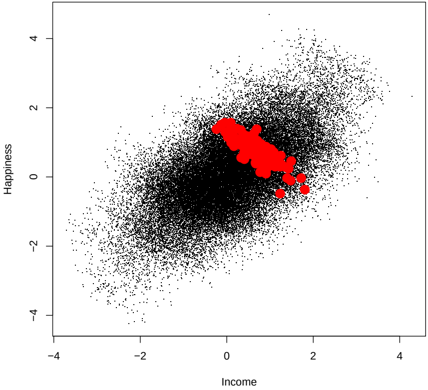
Modern Modeling Methods (M^3)
Storrs, CT; May 25, 2016

Outline

- Introduction, Background, and Motivation
- State Space Models and Kalman Filters
- Examples
  - Factor Models
  - Standard Structural Equation Models
  - Latent Growth Models
  - Cross-Lagged Panel Models
  - Dual Change Score Models
  - Autoregressive Latent Trajectory Models
  - ...
- Discussion, Conclusions, and Future Work

Perspective



Where to go from here?

- Between-person models are valid, but (generally) only between people.
- Conclusions for individuals require repeated measurements for individuals.
- Model individuals and processes.
- Balance the Idiographic/Nomothetic trade-off

Where to go from here?

- Between-person models are valid, but (generally) only between people.
- Conclusions for individuals require repeated measurements for individuals.
- Model individuals and processes.
- Balance the Idiographic/Nomothetic trade-off
- How do you model variability within people?

State Space Model

Measurement

- Structural Equation Measurement Model

$$\boldsymbol{y}_i = \Lambda \boldsymbol{\eta}_i + K \boldsymbol{x}_i + \boldsymbol{\varepsilon}_i \quad \text{with} \quad \boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}, \Theta) \tag{1}$$

- State Space Measurement Model

$$\boldsymbol{y}_i = \Lambda \boldsymbol{\eta}_i + K \boldsymbol{x}_i + \boldsymbol{\varepsilon}_i \quad \text{with} \quad \boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}, \Theta) \tag{2}$$

State Space Model

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- Example 1 is done.

State Space Model

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- Example 1 is done.
- OpenMx Notation

$$\boldsymbol{y}_i = C \boldsymbol{x}_i + D \boldsymbol{u}_i + \boldsymbol{r}_i \quad \text{with} \quad \boldsymbol{r}_i \sim \mathcal{N}\left(\mathbf{0}, R\right) \tag{3}$$

State Space Model

Transition/Structural

- Structural Equation Structural Model

$$\boldsymbol{\eta}_i = B\boldsymbol{\eta}_i + \Gamma \boldsymbol{x}_i + \boldsymbol{\zeta}_i \quad \text{with} \quad \boldsymbol{\zeta}_i \sim \mathcal{N}\left(\mathbf{0}, \Psi\right) \tag{4}$$

- State Space Structural Model

$$\boldsymbol{\eta}_i = B\boldsymbol{\eta}_{i-1} + \Gamma \boldsymbol{x}_i + \boldsymbol{\zeta}_i \quad \text{with} \quad \boldsymbol{\zeta}_i \sim \mathcal{N}\left(\mathbf{0}, \Psi\right) \tag{5}$$

State Space Model

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- OpenMx Notation

$$\boldsymbol{x}_i = A\boldsymbol{x}_{i-1} + B\boldsymbol{u}_i + \boldsymbol{q}_i \quad \text{with} \quad \boldsymbol{q}_i \sim \mathcal{N}\left(\mathbf{0}, Q\right) \tag{6}$$

State Space Model

OpenMx Notation

- Structure

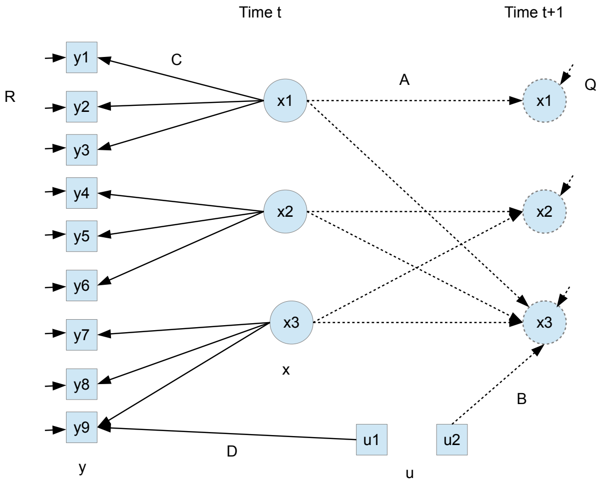
$$\boldsymbol{x}_i = A\boldsymbol{x}_{i-1} + B\boldsymbol{u}_i + \boldsymbol{q}_i \quad \text{with} \quad \boldsymbol{q}_i \sim \mathcal{N}\left(\mathbf{0}, Q\right) \tag{7}$$

- Measurement

$$\boldsymbol{y}_i = C\boldsymbol{x}_i + D\boldsymbol{u}_i + \boldsymbol{r}_i \quad \text{with} \quad \boldsymbol{r}_i \sim \mathcal{N}\left(\mathbf{0}, R\right) \tag{8}$$

State Space Model

Diagrams



State Space Model

Available now in OpenMx

- Differential Equation in Discrete Time
- Implemented by me in OpenMx 2.0 Release
- Continuous Time is in OpenMx 2.1 Release
- Current Release is 2.5.2/2.6.2

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}_i + \boldsymbol{q}(t) \tag{9}$$

- Uses Kalman filter

Kalman Filter Benefits

B-B-B-Benny and the Fits

- Designed for non-stationary time series
  - Cf. block-Toeplitz autocovariances (Molenaar, 1985)
  - Cf. lagged observed variables (Song & Zhang, 2014)
  - Cf. exact discrete model (Voelkle & Oud, 2013; Driver, Oud, & Voelkle, 2015)
- Gaussian noise: gives ML estimates
- Non-Gaussian noise: becomes least squares optimal
- Latent State Estimates are factor scores (Priestley & Subba Rao, 1975)
- Latent Covariace Estimates ⇒ Reliability (Hunter, In Preparation)

Kalman Filter

Equations

- Predict Step

$$\boldsymbol{x}_{t|t-1} = A\boldsymbol{x}_{t-1|t-1} + B\boldsymbol{u}_t \quad (10)$$

$$P_{t|t-1} = AP_{t-1|t-1}A^{\mathsf{T}} + Q \quad (11)$$

- Update Step

$$\widehat{\boldsymbol{y}}_t = \widehat{Mean}(\boldsymbol{y}_t) = C\boldsymbol{x}_{t|t-1} + D\boldsymbol{u}_t \quad (12)$$

$$\widetilde{\boldsymbol{y}}_t = \widehat{Residual}(\boldsymbol{y}_t) = \boldsymbol{y}_t - \widehat{\boldsymbol{y}}_t \quad (13)$$

$$\widehat{S}_t = \widehat{Cov}(\boldsymbol{y}_t) = CP_{t|t-1}C^{\mathsf{T}} + R \quad (14)$$

$$K = P_{t|t-1}C^{\mathsf{T}}\widehat{S}_t^{-1} \quad (15)$$

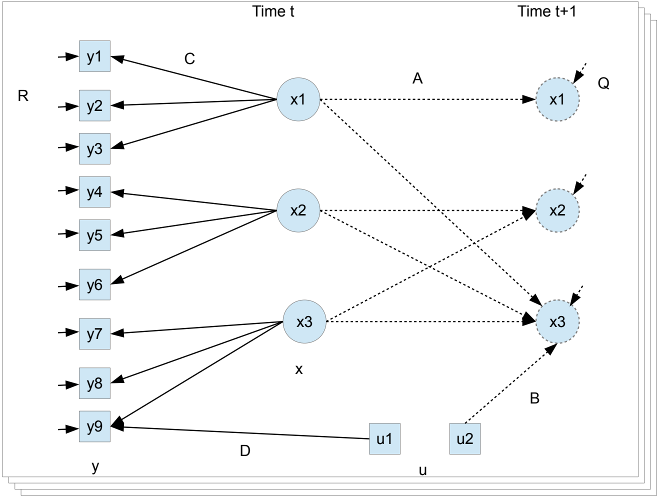
$$\boldsymbol{x}_{t|t} = \boldsymbol{x}_{t|t-1} + K\,\widetilde{\boldsymbol{y}}_t \quad (16)$$

$$P_{t|t} = P_{t|t-1} - KCP_{t|t-1} \quad (17)$$

What about multiple individuals?

State Space Model

Diagrams



The Factor Model?

That's a state space model.

The Standard Structural Equation Model?

That's a state space model.

SEM as a State Space Model

Math ...go figure

The standard structural equation model is a factor model.

$$\boldsymbol{\eta}_i = B\boldsymbol{\eta}_i + \Gamma \boldsymbol{x}_i + \boldsymbol{\zeta}_i \quad \text{with} \quad \boldsymbol{\zeta}_i \sim \mathcal{N}\left(\mathbf{0}, \Psi\right) \tag{18}$$

$$\boldsymbol{\eta}_i - B\boldsymbol{\eta}_i = (I - B)\boldsymbol{\eta}_i = \Gamma \boldsymbol{x}_i + \boldsymbol{\zeta}_i \quad \text{with} \quad \boldsymbol{\zeta}_i \sim \mathcal{N}\left(\mathbf{0}, \Psi\right) \tag{19}$$

$$\boldsymbol{\eta}_i = (I - B)^{-1}\Gamma \boldsymbol{x}_i + (I - B)^{-1}\boldsymbol{\zeta}_i \quad \text{with} \quad \boldsymbol{\zeta}_i \sim \mathcal{N}\left(\mathbf{0}, \Psi\right) \tag{20}$$

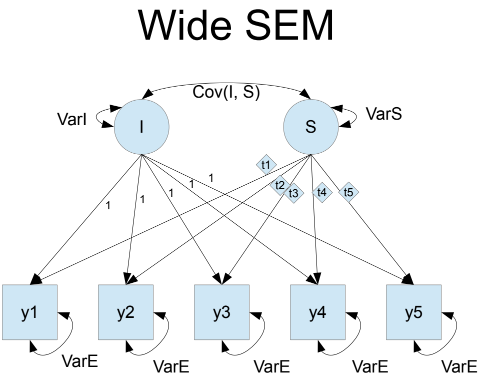
$$\boldsymbol{\eta}_i = \Gamma_2 \boldsymbol{x}_i + \boldsymbol{\zeta}_{2,i} \quad \text{with} \quad \boldsymbol{\zeta}_i \sim \mathcal{N}\left(\mathbf{0}, (I - B)^{-1}\Psi(I - B)^{-\top}\right) \tag{21}$$

The Latent Growth Model?

That's a state space model.

Latent Growth Curve

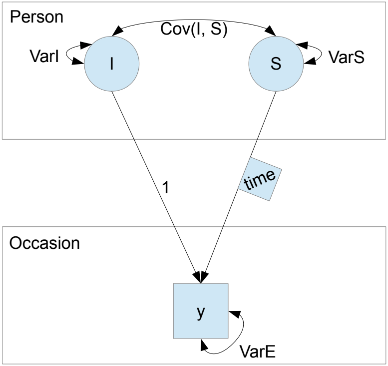
Tucker (1958) & Rao (1958)



Latent Growth Curve

Tucker (1958) & Rao (1958)

Relational SEM



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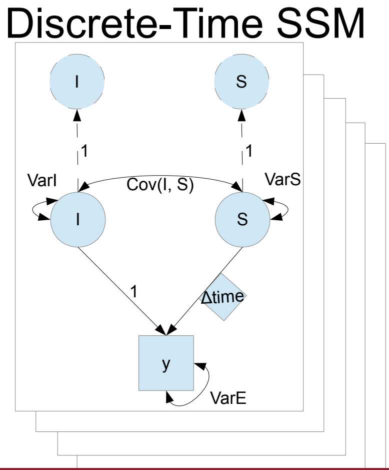
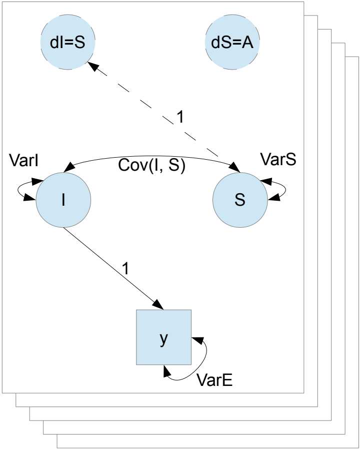


Latent Growth Curve

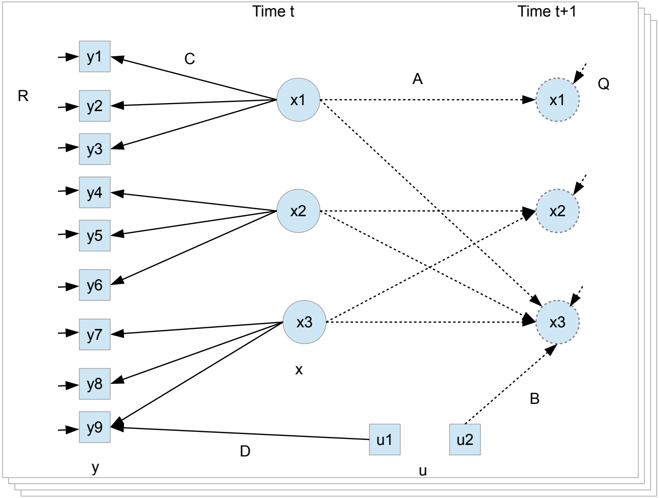
Continuous-Time SSM



The (Latent) Cross-Lagged Panel Model?

That's a state space model.

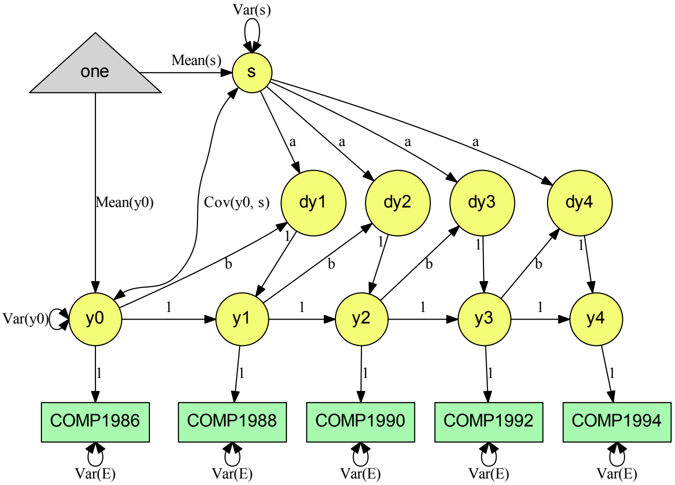
Cross-Lagged Panel Model



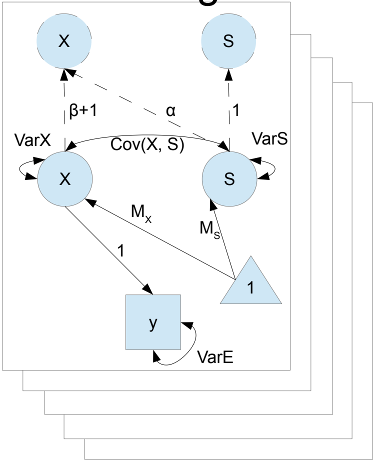
The Dual Change Score Model?

That's a state space model.

Dual Change Score

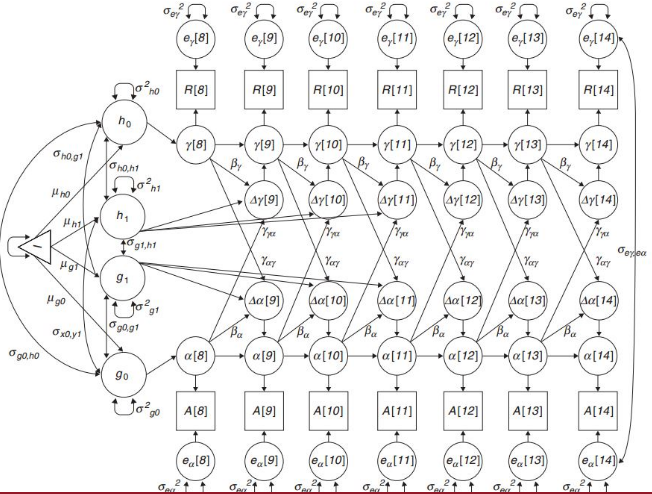


Dual Change SSM

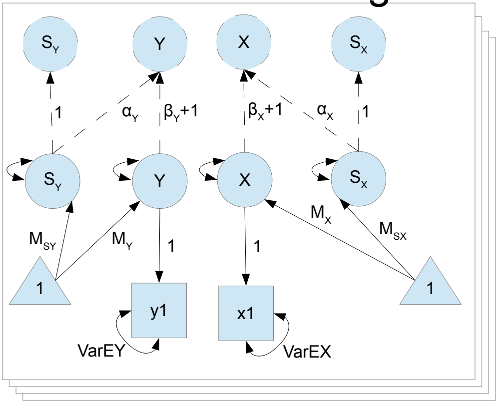


Bivariate Dual Change Score

McArdle & Grimm (2010) "Five Steps"



Bivariate Dual Change SSM

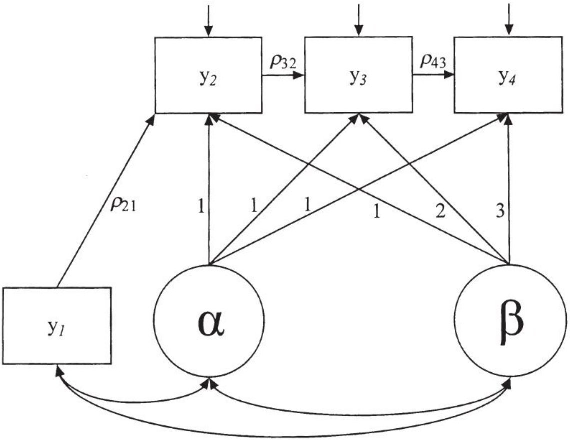


The Autoregressive Latent Trajectory Model?

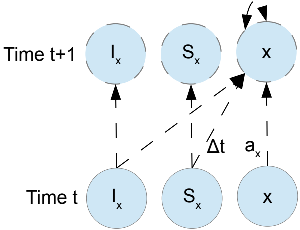
That's a state space model.

Univariate ALT

Bollen & Curran (2004)



State Space ALT



Bivariate ALT

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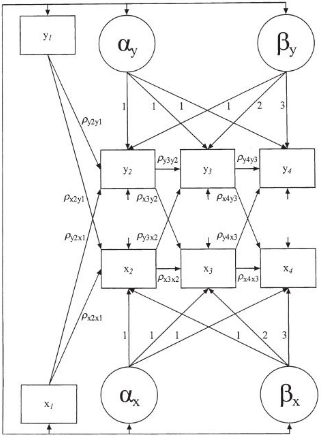
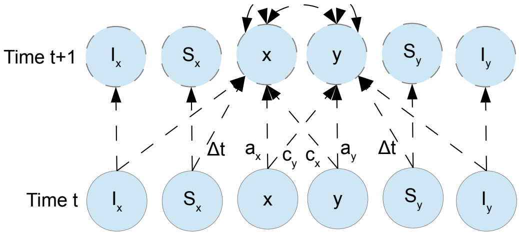


State Space Bivariate ALT



Why have I done this?

1. To see relationships among Models™
2. To have a common way of expressing temporal relationships

Bivariate Dual Change Score

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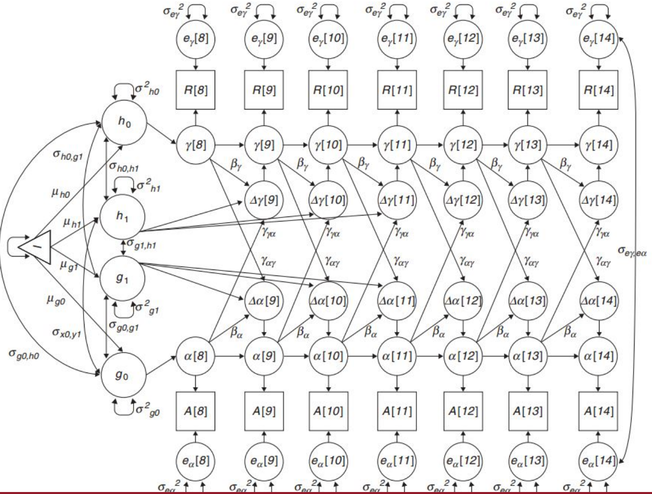

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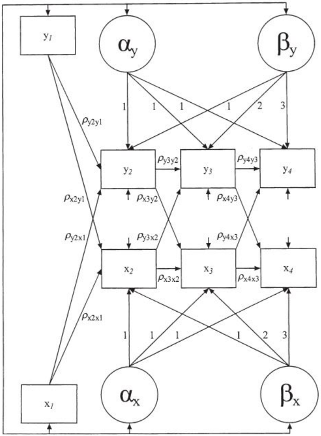
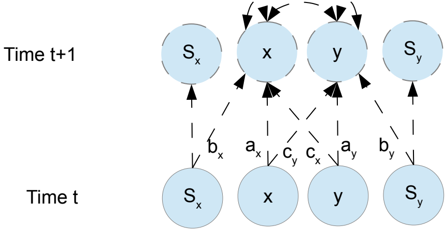
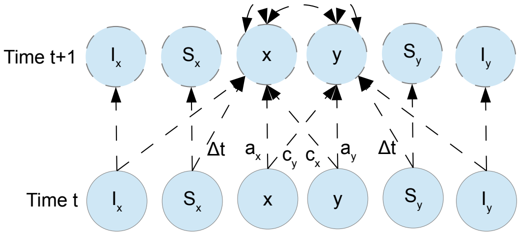


State Space Bivariate DCS



State Space Bivariate ALT



Summary

- Between- and Within-person variabilities are distinct
- Within-person models are needed for within-person conclusions
- The discrete-time linear state space model encompasses many models of change in a single framework.
  - Factor Models
  - Standard Structural Equation Models
  - Latent Growth Models
  - Cross-Lagged Panel Models
  - Dual Change Score Models
  - Autoregressive Latent Trajectory Models
  - ...
- Common language is the foundation of communication.

Future Work

- Emphasize continuous-time modeling for generalizable results.
- Integrate state space models with Item Factor Analysis.
- Integrate state space models with relational SEM.

Acknowledgments

- OpenMx Core Development Team
- Sy-Miin Chow
- Come see "What's for dynr?"

Thank You

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Driver, C. C., Oud, J. H. L., & Voelkle, M. C. (2015). Continuous time structural equation modelling with R package ctsem. *Journal of Statistical Software*, *77*(5). doi: 10.18637/jss.v077.i05

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