Path Analysis and Components of Covariance

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Introduction to SEM
Psyc–8501–001

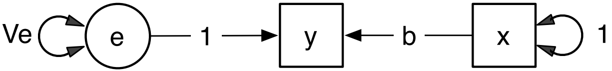
Path Analysis

- We can represent a structural model as a *path diagram*.
- Squares are manifest variables.
- Circles are latent variables.
- Triangles are constants.
- Single headed arrows are regression coefficients.
- Double headed arrows are variances and covariances.

Path Analysis

	Manifest Variable Measured Variable Predictor or Outcome Variable
	Latent Variable Unmeasured Variable Predictor or Outcome Variable
	Constant Always equal to 1 Used to extract means
	Variance of a variable Can be fixed or free If fixed to 1 then standardized
	Regression Coefficient Can be fixed or free Standardized or unstandardized
	Covariance Coefficient Can be fixed or free Standardized or unstandardized

Path Analysis: Standardized Univariate Regression



$$y = bx + e$$

$$\mathbf{R} = \begin{bmatrix} \mathrm{Var}(x) & \mathrm{Cov}(x, y) & \mathrm{Cov}(x, e) \\ \mathrm{Cov}(y, x) & \mathrm{Var}(y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x) & \mathrm{Cov}(e, y) & \mathrm{Var}(e) \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

- But note that the variance of x

$$\mathrm{Var}(x) = \sum_{i=1}^{N}(x - \bar{x})^2$$

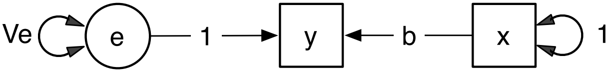
is the same as the covariance of x with x

$$\mathrm{Cov}(x,x) = \sum_{i=1}^{N}(x - \bar{x})(x - \bar{x})$$

- so, we can rewrite the covariance matrix $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} \mathrm{Cov}(x,x) & \mathrm{Cov}(x,y) & \mathrm{Cov}(x,e) \\ \mathrm{Cov}(y,x) & \mathrm{Cov}(y,y) & \mathrm{Cov}(y,e) \\ \mathrm{Cov}(e,x) & \mathrm{Cov}(e,y) & \mathrm{Cov}(e,e) \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression



$$y = bx + e$$

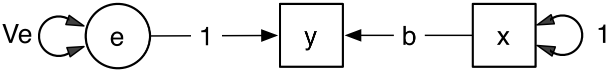
$$\mathbf{R} = \begin{bmatrix} \mathrm{Cov}(x,x) & \mathrm{Cov}(x,y) & \mathrm{Cov}(x,e) \\ \mathrm{Cov}(y,x) & \mathrm{Cov}(y,y) & \mathrm{Cov}(y,e) \\ \mathrm{Cov}(e,x) & \mathrm{Cov}(e,y) & \mathrm{Cov}(e,e) \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

- Since these are standardized variables, the variance of x and y are 1.0
- From the diagram, the covariance of e with e is V_e
- And by assumption, the covariance of e with x is zero
- so, let's substitute those into the covariance matrix $\mathbf{R}$

$$\mathbf{R} = \begin{bmatrix} 1.0 & \text{Cov}(x, y) & 0 \\ \text{Cov}(y, x) & 1.0 & \text{Cov}(y, e) \\ 0 & \text{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression



$$y = bx + e$$

$$\mathbf{R} = \begin{bmatrix} 1.0 & \text{Cov}(x,y) & 0 \\ \text{Cov}(y,x) & 1.0 & \text{Cov}(y,e) \\ 0 & \text{Cov}(e,y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

- Let's write our equation as a matrix equation, so that

$$y_i = bx_i + e_i$$

becomes

$$\mathbf{Y} = \mathbf{XB} + \mathbf{E}$$

where $\mathbf{X}$ is an $N \times 1$ matrix of scores with mean zero and

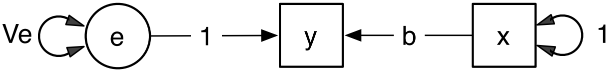
$$\mathbf{B} = \left[\begin{array}{c} b \end{array} \right]$$

and $\mathbf{E}$ is an $N \times 1$ matrix of residuals with mean zero.

Path Analysis: Standardized Univariate Regression

$$
\begin{aligned}
\mathbf{C_{YY}} &= 1/N(\mathbf{Y}'\mathbf{Y}) \\
&= 1/N((\mathbf{XB}+\mathbf{E})'(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(((\mathbf{XB})'+\mathbf{E}')(\mathbf{XB}+\mathbf{E})) \\
&= 1/N((\mathbf{B}'\mathbf{X}'+\mathbf{E}')(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(\mathbf{B}'\mathbf{X}'\mathbf{XB}+\mathbf{B}'\mathbf{X}'\mathbf{E}+\mathbf{E}'\mathbf{XB}+\mathbf{E}'\mathbf{E}) \\
&= 1/N(\mathbf{B}'\mathbf{X}'\mathbf{XB}+\mathbf{B}'0+0\mathbf{B}+\mathbf{E}'\mathbf{E}) \\
&= 1/N(\mathbf{B}'\mathbf{X}'\mathbf{XB}+\mathbf{E}'\mathbf{E}) \\
&= \mathbf{B}'(1/N\mathbf{X}'\mathbf{X})\mathbf{B}+1/N\mathbf{E}'\mathbf{E} \\
&= \mathbf{B}'\mathbf{C_{XX}}\mathbf{B}+\mathbf{C_{EE}} \\
&= b1b+V_e \\
1.0 &= b^2+V_e \\
1.0-V_e &= b^2 \\
r^2 &= b^2
\end{aligned}
$$

Path Analysis: Standardized Univariate Regression



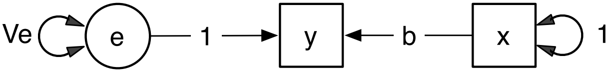
$$y = bx + e$$

$$\mathbf{R} = \begin{bmatrix} 1.0 & \text{Cov}(x, y) & 0 \\ \text{Cov}(y, x) & b^2 + V_e & \text{Cov}(y, e) \\ 0 & \text{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Univariate Regression

$$
\begin{aligned}
\mathbf{C_{XY}} &= 1/N(\mathbf{X}'\mathbf{Y}) \\
&= 1/N(\mathbf{X}'(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(\mathbf{X}'\mathbf{XB}+\mathbf{X}'\mathbf{E}) \\
&= 1/N(\mathbf{X}'\mathbf{XB}+0) \\
&= (1/N\mathbf{X}'\mathbf{X})\mathbf{B} \\
&= \mathbf{C_{XX}}\mathbf{B} \\
&= 1b \\
&= b
\end{aligned}
$$

Path Analysis: Standardized Univariate Regression



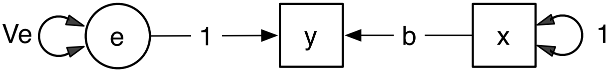
$$y = bx + e$$

$$\mathbf{R} = \begin{bmatrix} 1.0 & b & 0 \\ b & b^2 + V_e & \text{Cov}(y, e) \\ 0 & \text{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Univariate Regression

$$
\begin{aligned}
\mathbf{C_{EY}} &= 1/N(\mathbf{E}'\mathbf{Y}) \\
&= 1/N(\mathbf{E}'(\mathbf{XB} + \mathbf{E})) \\
&= 1/N(\mathbf{E}'\mathbf{XB} + \mathbf{E}'\mathbf{E}) \\
&= (1/N\mathbf{E}'\mathbf{X})\mathbf{B} + (1/N\mathbf{E}'\mathbf{E}) \\
&= 0\mathbf{B} + \mathbf{C_{EE}} \\
&= V_e
\end{aligned}
$$

Path Analysis: Standardized Univariate Regression



$$y = bx + e$$

$$\mathbf{R} = \begin{bmatrix} 1.0 & b & 0 \\ b & b^2 + V_e & V_e \\ 0 & V_e & V_e \end{bmatrix}$$

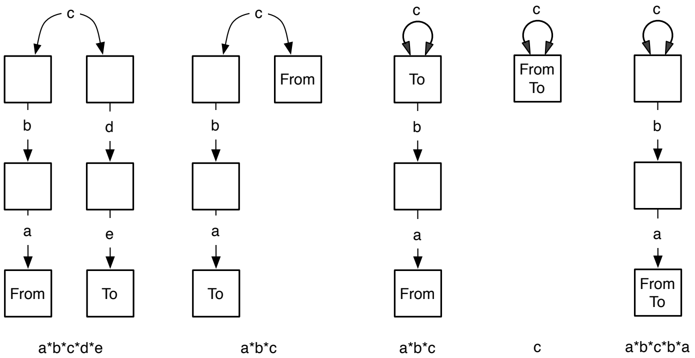
Path Analysis

- Now let's try it again.
- This time we'll use path tracing rules.
- These are the rules first created by Wright and amended by McArdle.

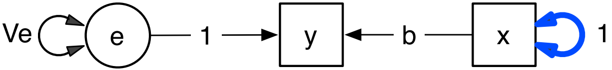
Path Analysis

- Pick any two variables in the diagram.
- Asymmetric paths are composed of **zero or more** one–headed arrows.
- Symmetric paths are composed **exactly one** two–headed arrows.
- For each pair of asymmetric paths that end in each of the chosen variables.
  - If there is a symmetric path connecting the beginning of the two asymmetric paths, multiply all the values on the one double headed arrow and all single headed arrows connecting the two chosen variables.

Path Analysis



Path Analysis: Standardized Univariate Regression

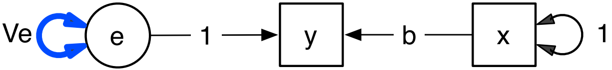


$$\mathrm{Cov}(x, x)$$

$$\begin{array}{ccc} x & \leftrightarrow & x \\ & 1 & \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & \mathrm{Cov}(x, y) & \mathrm{Cov}(x, e) \\ \mathrm{Cov}(y, x) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x) & \mathrm{Cov}(e, y) & \mathrm{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

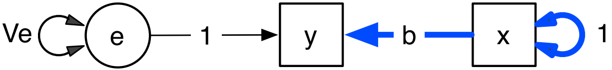


$$\text{Cov}(e, e)$$

$$\begin{array}{ccc} e & \leftrightarrow & e \\ & V_e & \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & \text{Cov}(x, y) & \text{Cov}(x, e) \\ \text{Cov}(y, x) & \text{Cov}(y, y) & \text{Cov}(y, e) \\ \text{Cov}(e, x) & \text{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

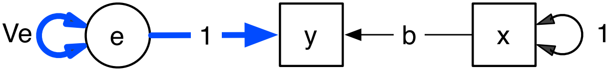


$$\mathrm{Cov}(x, y)$$

$$\begin{array}{ccccc} x & \leftrightarrow & x & \rightarrow & y \\ & 1 & \times & b & \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & b & \mathrm{Cov}(x,e) \\ b & \mathrm{Cov}(y,y) & \mathrm{Cov}(y,e) \\ \mathrm{Cov}(e,x) & \mathrm{Cov}(e,y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

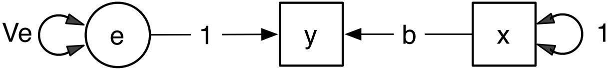


$$\text{Cov}(e, y)$$

$$\begin{array}{ccccc} e & \leftrightarrow & e & \rightarrow & y \\ & V_e & \times & 1 & \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & b & \text{Cov}(x, e) \\ b & \text{Cov}(y, y) & V_e \\ \text{Cov}(e, x) & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression

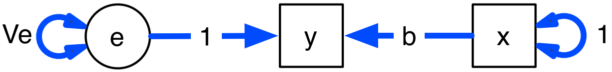


$$\text{Cov}(e, x)$$

no asymmetric paths point to either e or x

$$\mathbf{R} = \begin{bmatrix} 1 & b & 0 \\ b & \text{Cov}(y, y) & V_e \\ 0 & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Univariate Regression



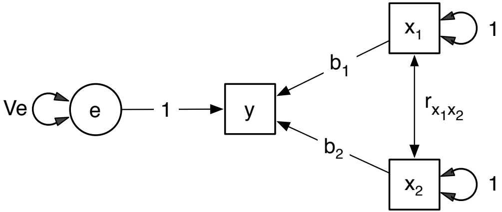
$$\text{Cov}(y, y)$$

$$\begin{array}{ccccccc} y & \leftarrow & x & \leftrightarrow & x & \rightarrow & y \\ & b & \times & 1 & \times & b & \end{array}$$

$$\begin{array}{ccccccc} y & \leftarrow & e & \leftrightarrow & e & \rightarrow & y \\ & 1 & \times & V_e & \times & 1 & \end{array}$$

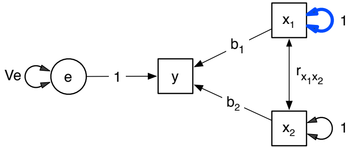
$$\mathbf{R} = \begin{bmatrix} 1 & b & 0 \\ b & b^2 + V_e & V_e \\ 0 & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression



$$\mathbf{R} = \begin{bmatrix} \mathrm{Cov}(x_1, x_1) & \mathrm{Cov}(x_1, x_2) & \mathrm{Cov}(x_1, y) & \mathrm{Cov}(x_1, e) \\ \mathrm{Cov}(x_2, x_1) & \mathrm{Cov}(x_2, x_2) & \mathrm{Cov}(x_2, y) & \mathrm{Cov}(x_2, e) \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x_1) & \mathrm{Cov}(e, x_2) & \mathrm{Cov}(e, y) & \mathrm{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

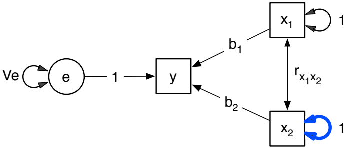


$$\mathrm{Cov}(x_1, x_1)$$

$$x_1 \quad \leftrightarrow \quad x_1$$
$$1$$

$$\mathbf{R} = \begin{bmatrix} 1 & \mathrm{Cov}(x_1, x_2) & \mathrm{Cov}(x_1, y) & \mathrm{Cov}(x_1, e) \\ \mathrm{Cov}(x_2, x_1) & \mathrm{Cov}(x_2, x_2) & \mathrm{Cov}(x_2, y) & \mathrm{Cov}(x_2, e) \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x_1) & \mathrm{Cov}(e, x_2) & \mathrm{Cov}(e, y) & \mathrm{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

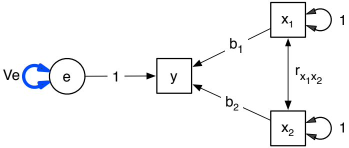


$$\text{Cov}(x_2, x_2)$$

$$x_2 \quad \leftrightarrow \quad x_2$$
$$1$$

$$\mathbf{R} = \begin{bmatrix} 1 & \text{Cov}(x_1, x_2) & \text{Cov}(x_1, y) & \text{Cov}(x_1, e) \\ \text{Cov}(x_2, x_1) & 1 & \text{Cov}(x_2, y) & \text{Cov}(x_2, e) \\ \text{Cov}(y, x_1) & \text{Cov}(y, x_2) & \text{Cov}(y, y) & \text{Cov}(y, e) \\ \text{Cov}(e, x_1) & \text{Cov}(e, x_2) & \text{Cov}(e, y) & \text{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

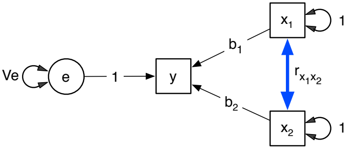


$$\text{Cov}(e, e)$$

$$\begin{matrix} e & \leftrightarrow & e \\ & V_e & \end{matrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & \text{Cov}(x_1, x_2) & \text{Cov}(x_1, y) & \text{Cov}(x_1, e) \\ \text{Cov}(x_2, x_1) & 1 & \text{Cov}(x_2, y) & \text{Cov}(x_2, e) \\ \text{Cov}(y, x_1) & \text{Cov}(y, x_2) & \text{Cov}(y, y) & \text{Cov}(y, e) \\ \text{Cov}(e, x_1) & \text{Cov}(e, x_2) & \text{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

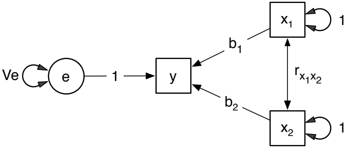


$$\mathrm{Cov}(x_1, x_2)$$

$$x_1 \quad \leftrightarrow \quad x_2$$
$$r_{x_1 x_2}$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & \mathrm{Cov}(x_1, y) & \mathrm{Cov}(x_1, e) \\ r_{x_1 x_2} & 1 & \mathrm{Cov}(x_2, y) & \mathrm{Cov}(x_2, e) \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x_1) & \mathrm{Cov}(e, x_2) & \mathrm{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

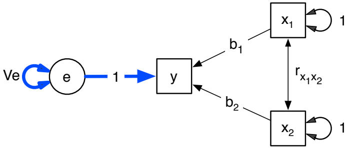


$$\mathrm{Cov}(e, x_1),\ \mathrm{Cov}(e, x_2)$$

no asymmetric paths point to e, x_1, or x_2

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & \mathrm{Cov}(x_1, y) & 0 \\ r_{x_1 x_2} & 1 & \mathrm{Cov}(x_2, y) & 0 \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ 0 & 0 & \mathrm{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

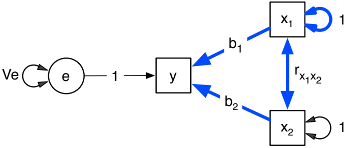


$$\mathrm{Cov}(e, y)$$

$$\begin{array}{ccccc} e & \leftrightarrow & e & \longrightarrow & y \\ & V_e & \times & 1 & \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1x_2} & \mathrm{Cov}(x_1, y) & 0 \\ r_{x_1x_2} & 1 & \mathrm{Cov}(x_2, y) & 0 \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & V_e \\ 0 & 0 & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

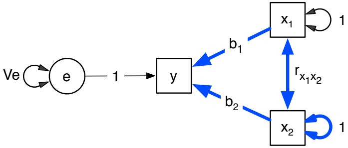


$$\mathrm{Cov}(x_1, y)$$

$$\begin{array}{ccccc}
x_1 & \leftrightarrow & x_1 & \rightarrow & y \\
 & 1 & \times & b_1 & \\
x_1 & \leftrightarrow & x_2 & \rightarrow & y \\
 & r_{x_1x_2} & \times & b_2 &
\end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1x_2} & b_1 + b_2 r_{x_1x_2} & 0 \\ r_{x_1x_2} & 1 & \mathrm{Cov}(x_2, y) & 0 \\ b_1 + b_2 r_{x_1x_2} & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & V_e \\ 0 & 0 & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

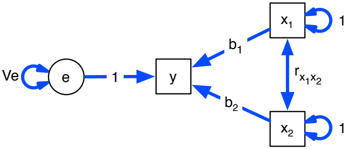


$$\mathrm{Cov}(x_2, y)$$

$$\begin{array}{ccccc} x_2 & \leftrightarrow & x_2 & \rightarrow & y \\ & 1 & \times & b_2 & \\ x_2 & \leftrightarrow & x_1 & \rightarrow & y \\ & r_{x_1 x_2} & \times & b_1 & \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & b_1 + b_2 r_{x_1 x_2} & 0 \\ r_{x_1 x_2} & 1 & b_2 + b_1 r_{x_1 x_2} & 0 \\ b_1 + b_2 r_{x_1 x_2} & b_2 + b_1 r_{x_1 x_2} & \mathrm{Cov}(y, y) & V_e \\ 0 & 0 & V_e & V_e \end{bmatrix}$$

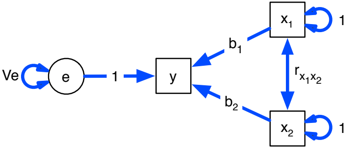
Path Analysis: Standardized Multiple Regression



$$\mathrm{Cov}(y, y)$$

$$\begin{array}{ccccccc}
y & \leftarrow & x_1 & \leftrightarrow & x_1 & \rightarrow & y \\
 & b_1 & \times & 1 & \times & b_1 & \\
y & \leftarrow & x_2 & \leftrightarrow & x_2 & \rightarrow & y \\
 & b_2 & \times & 1 & \times & b_2 & \\
y & \leftarrow & x_1 & \leftrightarrow & x_2 & \rightarrow & y \\
 & b_1 & \times & r_{x_1 x_2} & \times & b_2 & \\
y & \leftarrow & x_2 & \leftrightarrow & x_1 & \rightarrow & y \\
 & b_2 & \times & r_{x_1 x_2} & \times & b_1 & \\
y & \leftarrow & e & \leftrightarrow & e & \rightarrow & y \\
 & 1 & \times & V_e & \times & 1 &
\end{array}$$

Path Analysis: Standardized Multiple Regression



$$\mathrm{Cov}(y, y)$$

$$b_1^2 + b_2^2 + 2 b_1 b_2 r_{x_1 x_2} + V_e$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & b_1 + b_2 r_{x_1 x_2} & 0 \\ r_{x_1 x_2} & 1 & b_2 + b_1 r_{x_1 x_2} & 0 \\ b_1 + b_2 r_{x_1 x_2} & b_2 + b_1 r_{x_1 x_2} & b_1^2 + b_2^2 + 2 b_1 b_2 r_{x_1 x_2} + V_e & V_e \\ 0 & 0 & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

- Let's try the same thing using matrix algebra.

$$y_i = b_1 x_{i1} + b_2 x_{i2} + e_i$$

becomes

$$\mathbf{Y} = \mathbf{X}\mathbf{B} + \mathbf{E}$$

where $\mathbf{X}$ is an $N \times 2$ matrix of scores with mean zero and

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and $\mathbf{E}$ is an $N \times 1$ matrix of residuals with mean zero.

Path Analysis: Standardized Multiple Regression

$$y_i = b_1 x_{i1} + b_2 x_{i2} + e_i$$

$$\mathbf{R} = \begin{bmatrix} \mathrm{Cov}(x_1, x_1) & \mathrm{Cov}(x_1, x_2) & \mathrm{Cov}(x_1, y) & \mathrm{Cov}(x_1, e) \\ \mathrm{Cov}(x_2, x_1) & \mathrm{Cov}(x_2, x_2) & \mathrm{Cov}(x_2, y) & \mathrm{Cov}(x_2, e) \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x_1) & \mathrm{Cov}(e, x_2) & \mathrm{Cov}(e, y) & \mathrm{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

$$y_i \quad = \quad b_1 x_{i1} + b_2 x_{i2} + e_i$$

Note the the upper left 2×2 submatrix of $\mathbf{R}$ is $\mathbf{C_{XX}}$, where

$$\mathbf{C_{XX}} = \begin{bmatrix} 1 & r_{x_1 x_2} \\ r_{x_1 x_2} & 1 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & \mathrm{Cov}(x_1, y) & \mathrm{Cov}(x_1, e) \\ r_{x_1 x_2} & 1 & \mathrm{Cov}(x_2, y) & \mathrm{Cov}(x_2, e) \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ \mathrm{Cov}(e, x_1) & \mathrm{Cov}(e, x_2) & \mathrm{Cov}(e, y) & \mathrm{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

$$y_i = b_1 x_{i1} + b_2 x_{i2} + e_i$$

By assumption, the covariance between **X** and **E** is

$$\mathbf{C_{EX}} = 0$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1x_2} & \mathrm{Cov}(x_1, y) & 0 \\ r_{x_1x_2} & 1 & \mathrm{Cov}(x_2, y) & 0 \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ 0 & 0 & \mathrm{Cov}(e, y) & \mathrm{Cov}(e, e) \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

$$y_i = b_1 x_{i1} + b_2 x_{i2} + e_i$$

The covariance between **E** and **E** is given as

$$\mathbf{C}_{\mathbf{EE}} = V_e$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1x_2} & \mathrm{Cov}(x_1, y) & 0 \\ r_{x_1x_2} & 1 & \mathrm{Cov}(x_2, y) & 0 \\ \mathrm{Cov}(y, x_1) & \mathrm{Cov}(y, x_2) & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ 0 & 0 & \mathrm{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Univariate Regression

$$
\begin{aligned}
\mathbf{C_{XY}} &= 1/N(\mathbf{X}'\mathbf{Y}) \\
&= 1/N(\mathbf{X}'(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(\mathbf{X}'\mathbf{XB}+\mathbf{X}'\mathbf{E}) \\
&= 1/N(\mathbf{X}'\mathbf{XB}+0) \\
&= (1/N\mathbf{X}'\mathbf{X})\mathbf{B} \\
&= \mathbf{C_{XX}B} \\
&= \begin{bmatrix} 1 & r_{x_1x_2} \\ r_{x_1x_2} & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
&= \begin{bmatrix} b_1 + b_2 r_{x_1x_2} \\ b_1 r_{x_1x_2} + b_2 \end{bmatrix}
\end{aligned}
$$

Path Analysis: Standardized Multiple Regression

$$y_i = b_1 x_{i1} + b_2 x_{i2} + e_i$$

The covariance between **X** and **Y** reduced to

$$\mathbf{C_{XY}} = \begin{bmatrix} b_1 + b_2 r_{x_1 x_2} \\ b_1 r_{x_1 x_2} + b_2 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & b_1 + b_2 r_{x_1 x_2} & 0 \\ r_{x_1 x_2} & 1 & b_1 r_{x_1 x_2} + b_2 & 0 \\ b_1 + b_2 r_{x_1 x_2} & b_1 r_{x_1 x_2} + b_2 & \mathrm{Cov}(y, y) & \mathrm{Cov}(y, e) \\ 0 & 0 & \mathrm{Cov}(e, y) & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

$$
\begin{aligned}
\mathbf{C_{EY}} &= 1/N(\mathbf{E}'\mathbf{Y}) \\
&= 1/N(\mathbf{E}'(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(\mathbf{E}'\mathbf{XB}+\mathbf{E}'\mathbf{E}) \\
&= (1/N\mathbf{E}'\mathbf{X})\mathbf{B}+(1/N\mathbf{E}'\mathbf{E}) \\
&= 0\mathbf{B}+\mathbf{C_{EE}} \\
&= V_e
\end{aligned}
$$

Path Analysis: Standardized Multiple Regression

$$y_i \quad = \quad b_1 x_{i1} + b_2 x_{i2} + e_i$$

The covariance between **E** and **Y** reduced to $\mathbf{C_{EE}}$ where

$$\mathbf{C_{EE}} = V_e$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & b_1 + b_2 r_{x_1 x_2} & 0 \\ r_{x_1 x_2} & 1 & b_1 r_{x_1 x_2} + b_2 & 0 \\ b_1 + b_2 r_{x_1 x_2} & b_1 r_{x_1 x_2} + b_2 & \mathrm{Cov}(y, y) & V_e \\ 0 & 0 & V_e & V_e \end{bmatrix}$$

Path Analysis: Standardized Multiple Regression

$$
\begin{aligned}
\mathbf{C_{YY}} &= 1/N(\mathbf{Y}'\mathbf{Y}) \\
&= 1/N((\mathbf{XB}+\mathbf{E})'(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(((\mathbf{XB})'+\mathbf{E}')(\mathbf{XB}+\mathbf{E})) \\
&= 1/N((\mathbf{B}'\mathbf{X}'+\mathbf{E}')(\mathbf{XB}+\mathbf{E})) \\
&= 1/N(\mathbf{B}'\mathbf{X}'\mathbf{XB}+\mathbf{B}'\mathbf{X}'\mathbf{E}+\mathbf{E}'\mathbf{XB}+\mathbf{E}'\mathbf{E}) \\
&= 1/N(\mathbf{B}'\mathbf{X}'\mathbf{XB}+\mathbf{B}'0+0\mathbf{B}+\mathbf{E}'\mathbf{E}) \\
&= 1/N(\mathbf{B}'\mathbf{X}'\mathbf{XB}+\mathbf{E}'\mathbf{E}) \\
&= \mathbf{B}'(1/N\mathbf{X}'\mathbf{X})\mathbf{B}+1/N\mathbf{E}'\mathbf{E} \\
&= \mathbf{B}'\mathbf{C_{XX}}\mathbf{B}+\mathbf{C_{EE}} \\
&= \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} 1 & r_{x_1x_2} \\ r_{x_1x_2} & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + [V_e] \\
&= \begin{bmatrix} b_1^2 + b_2^2 + 2b_1b_2r_{x_1x_2} \end{bmatrix} + [V_e]
\end{aligned}
$$

Path Analysis: Standardized Multiple Regression

$$y_i \quad = \quad b_1 x_{i1} + b_2 x_{i2} + e_i$$

The covariance between **Y** and **Y** reduced to

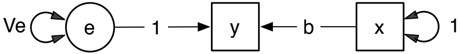
$$\mathbf{C_{YY}} = b_1^2 + b_2^2 + 2 b_1 b_2 r_{x_1 x_2} + V_e$$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{x_1 x_2} & b_1 + b_2 r_{x_1 x_2} & 0 \\ r_{x_1 x_2} & 1 & b_1 r_{x_1 x_2} + b_2 & 0 \\ b_1 + b_2 r_{x_1 x_2} & b_1 r_{x_1 x_2} + b_2 & b_1^2 + b_2^2 + 2 b_1 b_2 r_{x_1 x_2} + V_e & V_e \\ 0 & 0 & V_e & V_e \end{bmatrix}$$

RAM Algebra

- Wouldn't it be nice to be able to make this easier?
- RAM algebra allows you to specify the covariance expectations in a simple and general format.
- There are three RAM matrices.
- If there are p manifest variables and q total variables (manifest plus latent),
    1. The asymmetric matrix $\mathbf{A}$ is an $q \times q$ matrix where you specify all of the single headed arrows.
    2. The symmetric matrix $\mathbf{S}$ is an $q \times q$ matrix where you specify all of the double headed arrows.
    3. The filter matrix $\mathbf{F}$ is an $p \times q$ matrix where you specify which variables are manifest.

RAM Algebra: Standardized Univariate Regression

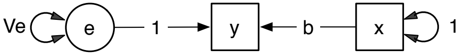


$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V_e \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

RAM Algebra: Standardized Univariate Regression



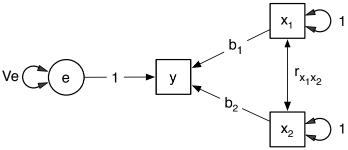
The effect matrix **E** is the total effects of all of the paths

$$\mathbf{E} = (\mathbf{I} - \mathbf{A})^{-1}$$

Then the expected covariances predicted by the model are

$$\begin{aligned}\mathbf{R} &= \mathbf{ESE}' \\ &= \begin{bmatrix} 1 & b & 0 \\ b & b^2 + V_e & V_e \\ 0 & V_e & V_e \end{bmatrix}\end{aligned}$$

RAM Algebra: Standardized Multiple Regression

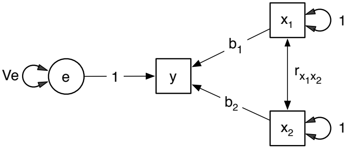


$$
\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\mathbf{S} = \begin{bmatrix} 1 & r_{x_1 x_2} & 0 & 0 \\ r_{x_1 x_2} & 1 & 0 & 0 \\ 0 & 0 & 0 & V_e \end{bmatrix}
$$

$$
\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

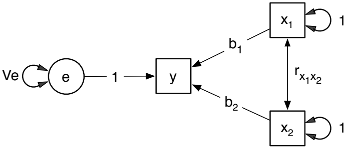
RAM Algebra: Standardized Multiple Regression



Then the expected covariances predicted by the model are

$$
\begin{aligned}
\mathbf{R} &= \mathbf{ESE}' \\
&= (\mathbf{I}-\mathbf{A})^{-1}\mathbf{S}((\mathbf{I}-\mathbf{A})^{-1})' \\
&= \begin{bmatrix}
1 & r_{x_1x_2} & b_1 + b_2 r_{x_1x_2} & 0 \\
r_{x_1x_2} & 1 & b_1 r_{x_1x_2} + b_2 & 0 \\
b_1 + b_2 r_{x_1x_2} & b_1 r_{x_1x_2} + b_2 & b_1^2 + b_2^2 + 2 b_1 b_2 r_{x_1x_2} + V_e & V_e \\
0 & 0 & V_e & V_e
\end{bmatrix}
\end{aligned}
$$

RAM Algebra: Standardized Multiple Regression



We can select just the covariances of the manifest variables by pre– and post–multiplying by the filter matrix

$$
\begin{aligned}
\mathbf{R} &= \mathbf{FESE'F'} \\
&= \mathbf{F}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{S}((\mathbf{I}-\mathbf{A})^{-1})'\mathbf{F}' \\
&= \begin{bmatrix} 1 & r_{x_1x_2} & b_1 + b_2 r_{x_1x_2} \\ r_{x_1x_2} & 1 & b_1 r_{x_1x_2} + b_2 \\ b_1 + b_2 r_{x_1x_2} & b_1 r_{x_1x_2} + b_2 & b_1^2 + b_2^2 + 2b_1 b_2 r_{x_1x_2} + V_e \end{bmatrix}
\end{aligned}
$$

Next Week

- Fitting manifest variable models with OpenMx.